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LETTER TO THE EDITOR

Pseudo-differential operators, BKP equation and Weistrass $P(x)$ function

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Abstract. We present a derivation of the BKP equation based on explicit computation with pseudo-differential operators. In the latter part of our analysis, we show that the stationary solutions of this equation are given by the Weistrass $P(x)$ function, by assuming that L^3 and L^5 commute with L^4. Nowhere have we used the machinery of algebraic geometry.

In a recent communication we have obtained some explicit solution of the BKP equation on a rational singular curve [1] where we used the elegant formalism of Krichver and Novikov. On the other hand it is known that nonlinear equations such as KP, BKP, etc, all possess various other kinds of solutions also. A special and important class of solutions can be extracted from the commutation property of differential and pseudo-differential operators [2]. In this respect it has been seen that the use of some Baker-Akhiezer function leads to some very general results about these solutions. But the machinery of algebraic geometry is not accessible to all. Hence in this communication we have adopted a very elementary approach to find out the structure of stationary solutions of BKP equation. In the first part of our letter we have shown how, by starting from a pseudo-differential operator, we can explicitly construct the BKP system, using only very elementary algebraic manipulation. In the next part we show that the assumption that $(L^4)_+$ commutes with both $(L^3)_+$ and $(L^5)_+$ leads to a class of stationary solutions of the BKP system in terms of the Weistrass elliptic function $P(x)$; doubly periodic on a torus. Here we have used the standard notation that L is the pseudo-differential operator; $\partial+u_0+u_1\partial^{-1}+u_2\partial^{-2}+\ldots$, and $(L^n)_+$ denotes the positive part of the nth power of L.

Let us start with the pseudo-differential operator [3]

$$L=\partial+\sum_{i=1}^{\infty} a_i\partial^{-1} \tag{1}$$

where we have omitted the constant-level term in L, where ∂^{-1} stands for the integral operator $\int_{-\alpha}^{x}$, and $(L)^n$ denotes the formal nth power of L, while $(L)^n_+$ denotes the corresponding part keeping only non-negative powers of ∂.

Now we get by a straightforward calculation

$$L^2=\partial^2+2a_1+\sum_{i=1}^{\infty} b_i\partial^{-1} \tag{2}$$

where

$$b_1 = a_1' + 2a_2 \qquad b_2 = a_2' + 2a_3 + a_1^2$$
$$b_3 = a_3' + 2a_4 + 2a_1a_2 - a_1a_1' \qquad \text{etc} \tag{3}$$

where

$$(L^2)_+ = \partial^2 + 2a_1 \tag{4}$$

$$L^3 = \partial^3 + 3a_1\partial + 3(a_1' + a_2) + \sum_{i=1}^{\infty} c_i\partial^{-1} \tag{5}$$

with

$$c_1 = a_1'' + 3a_2' + 3a_3 + 3a_1^2$$
$$c_2 = a_2'' + 3a_3' + 3a_4 + 6a_1a_2. \tag{6}$$

Also

$$(L^3)_+ = \partial^3 + 3a_1\partial + 3(a_1' + a_2). \tag{7}$$

Combining (2) and (5) we get

$$(L^5)_+ = \partial^5 + \sum_{i=0}^{3} d_i\partial^{-1} \tag{8}$$

$$d_0 = 10a_3' + 5a_4 + 20a_1a_2 + 20a_1a_1' + 5a_1''' + 10a_2''$$
$$d_1 = 10a_2' + 5a_3 + 10a_1^2 + 10a_1'' \tag{9}$$
$$d_2 = 10a_1' + 5a_2 \qquad d_3 = 5a_1$$

where we have quoted only the truncated form of the L^5 operator as the one with negative ∂^{-k} is not needed.

We now consider the following equations:

$$\frac{\partial L}{\partial y} = [(L^3)_+, L] \qquad \frac{\partial L}{\partial T} = [(L^5)_+, L]. \tag{10}$$

Using the above expressions for $(L^3)_+$, $(L^5)_+$, and L we get

$$[(L^3)_+, L] = (3a_3' - 2a_1''' + 6a_1a_1')\partial^{-1} + (3a_4' + 3a_3'' - a_1^{\text{iv}} - 6(a_1'^2 + a_1a_1''))\partial^{-2}$$
$$+ \{3a_5' + 3a_4'' + a_3''' + 9a_1'a_3 + 9a_1''a_1' + 3a_1a_1''' + 3a_1a_3'\}\partial^{-3} \tag{11}$$

and also

$$[(L^5)_+, L)] = (-4a_1^{\text{v}} + 10a_3''' + 10a_4'' + 5a_5' + 20a_1'a_3 + 30a_1^2a_1' - 10a_1a_1''' + 20a_3'a_1)\partial^{-1}. \tag{12}$$

Equations (10), (11) and (12) lead to

$$\frac{\partial a_1}{\partial y} = 3a_3' - 2a_1''' + 6a_1a_1'$$

$$\frac{\partial a_2}{\partial y} = 3a_4' + 3a_3'' - a_1^{\text{iv}} - 6(a_1^2 + a_1a_1'')$$

$$\frac{\partial a_1}{\partial t} = -4a_1^{\text{v}} + 10a_3''' + 10a_4'' + 5a_5' + 20a_1'a_3 + 30a_1^2a_1' - 10a_1a_1''' + 20a_3'a_1 \tag{13}$$

$$\frac{\partial a_3}{\partial y} = 3a_5' + 3a_4'' + a_3''' + 9a_1'a_3 + 9a_1''a_1' + 3a_1a_1''' + 3a_1a_3'$$

and so on.

Actually equations (10) lead to an infinite set of coupled nonlinear systems. To reproduce any single meaningful equation we must take recourse to a reduction procedure. In the present case we take recourse to the sixth-order reduction of L; in the sense of [4]. In short this means that L^6 will be a perfectly differential operator without any integral part. Furthermore we demand that L^6 will be of the following form:

$$L^6 = (\partial^3 + k_1\partial)^2 + k_2\partial^2 + (2k_2' + k_3)\partial + (k_2'' + k_4 + k_3') \tag{14}$$

$$\begin{aligned} L = {} & \partial + \tfrac{1}{3}k_1\,\partial^{-1} - \tfrac{1}{3}k_1'\,\partial^{-2} + \tfrac{1}{6}(\tfrac{4}{3}k_1'' + k_2 - \tfrac{2}{3}k_1^2)\partial^{-3} + \tfrac{1}{6}(k_3 - \tfrac{2}{3}k_1''' - \tfrac{1}{2}k_2' - \tfrac{5}{3}k_1k_1')\partial^{-4} \\ & + \tfrac{1}{6}(k_4 - 9k_3' - \tfrac{16}{6}\,k_2'' - \tfrac{37}{9}\,k_1^{0\mathrm{i}} - \tfrac{115}{9}\,k_1k_1'' + \tfrac{4}{3}k_1k_2 - \tfrac{4}{27}k_1^3)\partial^{-5} + \dots \end{aligned} \tag{15}$$

which implies $a_1' + a_2 = 0$.

It is interesting to note that this condition implies the absence of a constant-level term in $(L^3)_+$ (equation (5)). The same is also the case with $(L^5)_+$. So in general we may characterize the BKP hierarchy by the constraint that $(L^{2n+1})_+$ will not contain any constant-level term.

Therefore equation (13) leads to

$$a_4 = a_1''' - 2a_3' + k_1. \tag{16}$$

If we now set $a_1 = 2\omega_x$ then

$$a_3 = \tfrac{1}{3}\{2\omega_y + 4\omega''' - 12\omega_x^2 + k_2\} \tag{17}$$

and we also observe that equation (15) implies

$$a_4 = -\tfrac{2}{3}\omega'''' - \tfrac{4}{3}\omega_{xy} + 16\omega_x\omega_{xx} + k_1$$

whence on choosing $k_1 = k_2 = 0$, ω is seen to satisfy the BKP equation

$$\frac{\partial}{\partial x}(\omega_{xxxxx} + 30\omega_x\omega_{xx} - 5\omega_{xxy} - 30\omega_x\omega_y + 60\omega_x^3 + 9\omega_t) - 5\omega_{yy} = 0. \tag{18}$$

We emphasize that on assuming commutativity of $(L^3)_+$ and $(L^5)_+$ with $(L^4)_+$, we can generate stationary solutions of the BKP equation. In the previous communication [1] we obtained some typical solutions of the BKP equation on a singular rational algebraic curve. Let us now compute $[(L^3)_+, (L^4)_+]$, which turns out to be

$$\begin{aligned} & (2a_1''' - 3a_3' - 6a_1a_1')\partial^2 + (a_1^{\mathrm{iv}} + 6a_1a_1'' + 6a_1'^2) \\ & \quad + \{a_1^{\mathrm{v}} + 9(a_1a_1''' + a_1'a_1'') + 9a_1'a_1'' + 18a_1^2a_1'\}. \end{aligned} \tag{19}$$

Equating to zero coefficients of ∂^2, ∂ and constant term we obtain after trivial integration;

$$a_3 = \tfrac{1}{3}(2a_1'' - 3a_1^2) + c_1 \tag{20a}$$

$$a_1''' + 6a_1a_1' = c_2 \tag{20b}$$

$$a_1^{\mathrm{iv}} + 9a_1a_1'' + \tfrac{9}{2}a_1'^2 + 6a_1^3 = c_3 \tag{20c}$$

where c_1, c_2, c_3 are constants of integration. Now (20*b*) and (20*c*) imply,

$$a_1'' + 3a_1^2 = c_2x + c_4. \tag{21}$$

If we assume $c_4 = 0$ and scale a_1 and x then (21) can be rewritten as;

$$\frac{\partial^2 b_1}{\partial z^2} = 6b_1^2 + z \tag{22}$$

where $x=\sqrt{2}z$ and $b_1=-a_1$. Equation (21) is nothing but the canonical equation of type I in the Painléve classification as described by Ince [5]. On the other hand if we set $c_2=0$ and $c_4\neq 0$ then we deduce,

$$\left(\frac{\partial b_1}{\partial z}\right)^2=-4c_4b_1+4b_1^3-\tfrac{4}{3}c_3 \tag{23}$$

which is the equation satisfied by the Weistrass $P(z)$ function. It is actually an elliptic function doubly periodic on a torus.

In our above analysis we have shown how the BKP equation can be deduced from the algebra of pseudo-differential operators in a simple way and stationary solutions are also deduced from the commutativity properties of $(L)_+^M$ operators in terms of the Weistrass $P(x)$ function.

It may be mentioned that a similar analysis was performed for the KP equation by Grünbaum [6] and Lathaw [7]. We are grateful to the referee for bringing these references to our attention.

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